ON THE VARIATIONAL METHODS IN THE THEORY OF CREEP

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In solving a number of problems of the theory of creep (for instance, the problems of stability) by the variational methods, it turns out to be more convenient to vary simultaneously the stresses and displacements or the stress-rates and displacement-rates. In the theory of creep, the variational method based on varying the stresses and displacements was first proposed by Rosenblium [1] for the study of stability of a longitudinally compressed bar under conditions of creep. Here, the stresses and displacements which satisfy the equilibrium equations and given boundary conditions were varied. Later, in [2] for the creep of a body according to the flow theory [3], a variational equation based on varying the stress-rates and displacement-rates was proposed.

Shown below is a variational equation for the creep of a body according to the flow theory [3] with extensions and shears small as compared to unity [4], based on varying the stresses and displacements which satisfy the equilibrium equations, as well as those obtained from the equilibrium equations by differentiation with respect to time (for the sake of brevity the latter equations are referred to as the rate equilibrium equations).

1. At a fixed instant of time t the actual stresses σ_{jk} and displacements u_i are related to one another and also to the stress-rates and displacement-rates by the equations [3,4]

$$[(\delta_{ik} + u_{i,k})\,\sigma_{jk}]_{,j} = 0, \qquad [(\delta_{ik} + u_{i,k})\,\dot{\sigma}_{jk} + \dot{u}_{i,k}\sigma_{jk}]_{,j} = 0$$
$$T_i = (\delta_{ik} + u_{i,k})\,\sigma_{jk}n_j, \qquad \dot{T}_i = [(\delta_{ik} + u_{i,k})\,\dot{\sigma}_{jk} + \dot{u}_{i,k}\sigma_{jk}]\,n_j \qquad (1.1)$$

$$\dot{\varepsilon}_{jk} = \partial W / \partial \sigma_{jk}, \qquad \dot{\varepsilon}_{jk} = {}^1/_2\,(\dot{u}_{j,k} + \dot{u}_{k,j} + \dot{u}_{i,j}u_{i,k} + \dot{u}_{i,k}u_{i,j}) \qquad (1.2)$$
$$W = \Lambda + \partial \Pi / \partial t \qquad (i, k, j = 1, 2, 3)$$

Here, the Lagrangean curvilinear coordinate system is used, with body forces assumed to be absent, δ_{ik} is the Kronecker delta, the dot designates differentiation with respect to time, T_i is the projection of the module of surface tractions in the deformed state on the ith coordinate axis of the undeformed state; n_j is the cosine of the angle which the outward normal to the surface of the body before deformation makes with the jth coordinate axis of the undeformed state; Λ is a function of stresses and time; Π is a function of stresses (the energy of elastic deformations).

2. Let us compare, at a fixed instant of time t, the actual state of stress and deformation with another one which is characterized by the same stress-rates and displacement-rates but has different stresses and displacements, namely the stresses $\sigma_{jk} + \delta\sigma_{jk}$ and displacements $u_i + \delta u_i$, which are infinitely close to the actual ones and which satisfy the equilibrium equations and the rate equilibrium equations. These stresses and displacements will be called admissible.

Substituting the admissible stresses and displacements into the equilibrium equations and the rate equilibrium equations, and keeping infinitesimals of the first order only, we find that the variations of stresses and displacements must satisfy the equations

$$[(\delta_{ik} + u_{i,k})\,\delta\sigma_{jk} + \sigma_{jk}\delta u_{i,k}]_{,j} = 0, \qquad [\dot{\sigma}_{j,k}\delta u_{i,k} + \dot{u}_{i,k}\delta\sigma_{jk}]_{,j} = 0 \tag{2.1}$$

According to (1.2), the variations of the strain-rates are related to the variations of the deformations by the formula

$$\delta\dot{\varepsilon}_{jk} = {}^1/_2\,(\dot{u}_{i,j}\delta u_{i,k} + \dot{u}_{i,k}\delta u_{i,j}) \tag{2.2}$$

3. Since the actual stresses and displacements satisfy the equilibrium equations, it follows that [4, Chap. 3]:

$$\int_V \sigma_{jk}\dot{\varepsilon}_{jk}\,dV = \int_S T_i\dot{u}_i\,dS$$

Here, V, S are the volume and surface of the body before deformation. The admissible stresses and displacements also satisfy the equilibrium equations. Therefore

$$\int_V (\sigma_{jk} + \delta\sigma_{jk})\,(\dot{\varepsilon}_{jk} + \delta\dot{\varepsilon}_{jk})\,dV = \int_S (T_i + \delta T_i)\,\dot{u}_i\,dS \tag{3.1}$$

Omitting the second-order infinitesimals we find

$$\int_V \sigma_{jk}\delta\dot{\varepsilon}_{jk}dV + \int_V \dot{\varepsilon}_{jk}\delta\sigma_{jk}dV - \delta\int_S T_i\dot{u}_i dS = 0 \tag{3.2}$$

Formula (3.2) can be readily verified through direct integration by parts utilizing relations (2.2) and equations (2.1).

4. It follows from (1.2) that the second term in (3.2) is the variation of the integral

$$\int_V W\, dV$$

Let us show that the first term in (3.2) also is the variation of a certain expression. Integrating the second of equations (2.1) multiplied by u_i over the volume of the body we find

$$\int_V (\dot{u}_{i,k}\, \delta\sigma_{jk} + \dot{\sigma}_{jk}\delta u_{i,k})\, u_{i,j} dV = \int_S (\dot{u}_{i,k}\delta\sigma_{jk} + \dot{\sigma}_{jk}\delta u_{i,k})\, u_i n_j\, dS \qquad (4.1)$$

With the use of (1.1), (2.2) and (4.1), the first term in (3.2) can be written in the form

$$\int_V \sigma_{jk}\delta\dot{e}_{jk}\, dV = \frac{1}{2}\,\delta\,\frac{d}{dt}\int_V \sigma_{jk}u_{i,k}u_{i,j}dV - \int_S u_i\delta\dot{T}_i dS \qquad (4.2)$$

Let S_u, S_σ, $S_{\sigma u}$ be the parts of the surface $S (S = S_u + S_\sigma + S_{\sigma u})$, on which, respectively, the displacements, the surface tractions and the mixed boundary conditions are given. Then

$$\int_S u_i\delta\dot{T}_i dS = \delta\left\{\int_{S_u} \bar{u}_i\dot{T}_i dS + \int_{S_\sigma} (\dot{T}_i - \dot{\bar{T}}_i)\, u_i dS + \int_{S_{\sigma u}} [\bar{u}_\gamma\dot{T}_\gamma + (\dot{T}_\nu - \dot{\bar{T}}_\nu)\, u_\nu]\, dS\right\} \qquad (4.3)$$

Here, the given values of the components of displacements and rates of surface tractions are designated by bars. It follows from (4.2) and (4.3) that the first term in (3.2) is a total variation, so that (3.2) takes the form

$$\delta\Phi = \delta\left\{\int_V \left[W + \frac{1}{2}\frac{d}{dt}(\sigma_{jk}u_{i,k}u_{i,j})\right] dV - \int_S T_i\dot{u}_i dS - \right.$$
$$\left. - \int_{S_u} \bar{u}_i\dot{T}_i dS - \int_{S_\sigma} (\dot{T}_i - \dot{\bar{T}}_i)\, u_i dS - \int_{S_{\sigma u}} \left[\, \bar{u}_\gamma\dot{T}_\gamma + (\dot{T}_\nu - \dot{\bar{T}}_\nu)\, u_\nu\right] dS\right\} = 0 \qquad (4.4)$$

Among the stresses and displacements which satisfy the equilibrium equations and the rate equilibrium equations, the true distribution of stresses and displacements is characterized by the functional Φ being stationary.

When not only the extensions and shears are small as compared to unity, but also the angles of rotation [4], the displacements do not appear in the equilibrium equations and hence are not varied; the rates of surface tractions are expressed in terms of stress-rates only, and consequently are not varied either, and the variational equation (4.4) transforms into the one given by Kachanov [3]

$$\delta\Phi = \delta\left\{\int_V W\,dV - \int_S T_i \dot{u}_i dS\right\} = 0$$

If for arbitrary magnitudes of extensions and shears the relation between the strain-rates and the generalized stresses σ_{jk}^* (see [4, Chap. 2, Sec. 7]) can be taken in the form

$$\dot{\varepsilon}_{jk} = \partial W / \partial \sigma_{jk}^*$$

then the variational equation (4.4) is valid also for arbitrary magnitudes of extensions and shears. In that case in (4.4) σ_{jk} should be interpreted as generalized stresses.

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